

MATRICES OVER ZHOU NIL-CLEAN RINGS

MARJAN SHEIBANI AND HUANYIN CHEN

ABSTRACT. A ring R is Zhou nil-clean if every element in R is the sum of two tripotents and a nilpotent that commute. Let R be a Zhou nil-clean ring. If R is 2-primal (of bounded index), we prove that every square matrix over R is the sum of two tripotents and a nilpotent. These provides a large class of rings over which every square matrix has such decompositions by tripotent and nilpotent matrices.

1. INTRODUCTION

Throughout, all rings are associative with an identity. A ring R is strongly nil-clean provided that every element in R is the sum of an idempotent and a nilpotent that commute (see [5]). A ring R is strongly weakly nil-clean provided that every element in R is the sum or difference of a nilpotent and an idempotent that commute (see [3]). An element a in a ring R is tripotent if $a = a^3$. A ring R is strongly 2-nil-clean if every element in R is the sum of a tripotent and a nilpotent that commute (see [2]). The subjects of strongly and strongly weakly nil-clean rings are interested for so many mathematicians, e.g., [1, 4, 8, 9] and [11].

Very recently, Zhou investigated a class of rings in which elements are the sum of two tripotents and a nilpotent that commute (see [12]). For the convenience, we call such ring a Zhou nil-clean ring. The purpose of this paper is to explore matrices over Zhou nil-clean rings. A ring R is 2-primal if its Baer-McCoy radical (i.e., it is equal to the intersection of all prime ideals in R) coincides with the set of nilpotents in R . For instances, every commutative

2010 *Mathematics Subject Classification.* 16S34, 16U99, 16E50.

Key words and phrases. tripotent matrix; nilpotent matrix; Zhou ring; Zhou nil-clean ring.

(reduced) ring is 2-primal. A ring R is of bounded index if there exists $m \in \mathbb{N}$ such that $x^m = 0$ for all nilpotent $x \in R$. Let R be Zhou nil-clean. If R is 2-primal (of bounded index), we prove that every square matrix over R is the sum of two tripotents and a nilpotent. These provides a large class of rings over which every square matrix has such decompositions by tripotent and nilpotent matrices.

We use $N(R)$ to denote the set of all nilpotent elements in R and $J(R)$ the Jacobson radical of R . \mathbb{N} stands for the set of all natural numbers.

2. ZHOU RINGS

A ring R is a Zhou ring if every element in R is the sum of two tripotents that commute. The structure of Zhou rings was studied in [10]. This section is devoted to preliminary observations concerning on matrices over Zhou rings which will be used in the sequel. We begin with

Lemma 2.1. *Every square matrix over \mathbb{Z}_3 is the sum of two idempotents and a nilpotent.*

Proof. As every matrix over a field has a Frobenius normal form,

$$\text{we may assume that } A = \begin{pmatrix} 0 & & & c_0 \\ 1 & 0 & & c_1 \\ & 1 & 0 & c_2 \\ & & \ddots & \vdots \\ & & \ddots & 0 & c_{n-2} \\ & & & 1 & c_{n-1} \end{pmatrix}.$$

Case I. $c_{n-1} = 0$. Choose

$$W = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & 0 \\ & 1 & 0 & 0 \\ & & \ddots & \vdots \\ & & \ddots & 0 & 0 \\ & & & 1 & 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & & & c_0 \\ 0 & 0 & & c_1 \\ & 0 & 0 & c_2 \\ & & \ddots & \vdots \\ & & \ddots & 0 & c_{n-2} \\ & & & 0 & 1 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & & & 0 \\ 0 & 0 & & 0 \\ & 0 & 0 & 0 \\ & & \ddots & \vdots \\ & & \ddots & 0 & 0 \\ & & & 0 & -1 \end{pmatrix}.$$

Then $E_1^3 = E_1$ and $E_2^3 = E_2$, and so $A = E_1 + E_2 + W$.

Case II. $c_{n-1} = 1$. Choose

$$W = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & 0 \\ & 1 & 0 & 0 \\ & & \ddots & \vdots \\ & & \ddots & 0 & 0 \\ & & & 1 & 0 \end{pmatrix}, E = \begin{pmatrix} 0 & & & c_0 \\ 0 & 0 & & c_1 \\ & 0 & 0 & c_2 \\ & & \ddots & \vdots \\ & & \ddots & 0 & c_{n-2} \\ & & & 0 & 1 \end{pmatrix}.$$

Then $E^3 = E$, and so $A = E + 0 + W$.

Case III. $c_{n-1} = 2$. Choose

$$W = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & 0 \\ & 1 & 0 & 0 \\ & & \ddots & \vdots \\ & & \ddots & 0 & 0 \\ & & & 1 & 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & & & c_0 \\ 0 & 0 & & c_1 \\ & 0 & 0 & c_2 \\ & & \ddots & \vdots \\ & & \ddots & 0 & c_{n-2} \\ & & & 0 & 1 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & & & 0 \\ 0 & 0 & & 0 \\ & 0 & 0 & 0 \\ & & \ddots & \vdots \\ & & \ddots & 0 & 0 \\ & & & 0 & 1 \end{pmatrix}.$$

Then $E_1^3 = E_1$ and $E_2^3 = E_2$, and so $A = E_1 + E_2 + W$. \square

Lemma 2.2. *Every square matrix over \mathbb{Z}_5 is the sum of two tripotents and a nilpotent.*

Proof. As every matrix over a field has a Frobenius normal form,

we may assume that $A = \begin{pmatrix} 0 & & & c_0 \\ 1 & 0 & & c_1 \\ & 1 & 0 & c_2 \\ & & \ddots & \vdots \\ & & \ddots & 0 & c_{n-2} \\ & & & 1 & c_{n-1} \end{pmatrix}.$

Case I. $c_{n-1} = 0$. Choose

$$W = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & 0 \\ & 1 & 0 & 0 \\ & & \ddots & \vdots \\ & & \ddots & 0 & 0 \\ & & & 1 & 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & & & c_0 \\ 0 & 0 & & c_1 \\ & 0 & 0 & c_2 \\ & & \ddots & \vdots \\ & & \ddots & 0 & c_{n-2} \\ & & & 0 & 1 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & & & 0 \\ 0 & 0 & & 0 \\ & 0 & 0 & 0 \\ & & \ddots & \vdots \\ & & \ddots & 0 & 0 \\ & & & 0 & -1 \end{pmatrix}.$$

Then $E_1^3 = E_1$ and $E_2^3 = E_2$, and so $A = E_1 + E_2 + W$.

Case II. $c_{n-1} = 1$. Choose

$$W = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & 0 \\ & 1 & 0 & 0 \\ & & \ddots & \vdots \\ & & \ddots & 0 & 0 \\ & & & 1 & 0 \end{pmatrix}, E = \begin{pmatrix} 0 & & & c_0 \\ 0 & 0 & & c_1 \\ & 0 & 0 & c_2 \\ & & \ddots & \vdots \\ & & \ddots & 0 & c_{n-2} \\ & & & 0 & 1 \end{pmatrix}.$$

Then $E^3 = E$, and so $A = E + 0 + W$.

Case III. $c_{n-1} = -1$. Choose

$$W = \begin{pmatrix} 0 & & & & 0 \\ 1 & 0 & & & 0 \\ & 1 & 0 & & 0 \\ & & \ddots & & \vdots \\ & & \ddots & 0 & 0 \\ & & & 1 & 0 \end{pmatrix}, E = \begin{pmatrix} 0 & & & & c_0 \\ 0 & 0 & & & c_1 \\ & 0 & 0 & & c_2 \\ & & \ddots & & \vdots \\ & & \ddots & 0 & c_{n-2} \\ & & & 0 & -1 \end{pmatrix}.$$

Then $E^2 = -E$, and so $E^3 = E$ and $A = E + 0 + W$.

Case IV. $c_{n-1} = 2$. Choose

$$W = \begin{pmatrix} 0 & & & & 0 \\ 1 & 0 & & & 0 \\ & 1 & 0 & & 0 \\ & & \ddots & & \vdots \\ & & \ddots & 0 & 0 \\ & & & 1 & 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & & & & c_0 \\ 0 & 0 & & & c_1 \\ & 0 & 0 & & c_2 \\ & & \ddots & & \vdots \\ & & \ddots & 0 & c_{n-2} \\ & & & 0 & 1 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & & & & 0 \\ 0 & 0 & & & 0 \\ & 0 & 0 & & 0 \\ & & \ddots & & \vdots \\ & & \ddots & 0 & 0 \\ & & & 0 & 1 \end{pmatrix}.$$

Then $E_1^3 = E_1$ and $E_2^3 = E_2$, and so $A = E_1 + E_2 + W$.

Case IV. $c_{n-1} = -2$. Choose

$$W = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & 0 \\ & 1 & 0 & 0 \\ & & \ddots & \vdots \\ & & \ddots & 0 & 0 \\ & & & 1 & 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & & & c_0 \\ 0 & 0 & & c_1 \\ & 0 & 0 & c_2 \\ & & \ddots & \vdots \\ & & \ddots & 0 & c_{n-2} \\ & & & 0 & -1 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 0 & & & 0 \\ 0 & 0 & & 0 \\ & 0 & 0 & 0 \\ & & \ddots & \vdots \\ & & \ddots & 0 & 0 \\ & & & 0 & -1 \end{pmatrix}.$$

Then $E_1^3 = E_1$ and $E_2^3 = E_2$, and so $A = E_1 + E_2 + W$.

Therefore we complete the proof. \square

Recall that a ring R is a Yaqub ring if it is the subdirect product of \mathbb{Z}_3 's. A ring R is a Bell ring if it is the subdirect product of \mathbb{Z}_5 's. We have

Lemma 2.3. *Every Zhou ring is isomorphic to a strongly nil-clean ring of bounded index, a Yaqub ring, a Bell ring or products of such rings.*

Proof. Let R be a Zhou ring. In view of [10, Theorem 5.2], R is a ring R_1 , a Yaqub ring R_2 , a Bell ring R_3 or products of such rings. Here, $R_1/J(R_1)$ is Boolean and $U(R_1)$ is a group of exponent 2. Let $x \in J(R_1)$. By the proof of [10, Theorem 5.2], $x^6 = x^4$, and so $x^4 = 0$. Thus, $J(R)^4 = 0$. It follows by [9, Theorem 2.7], R_1 is a strongly nil-clean ring of bounded index 4. \square

Theorem 2.4. *Let R be a Zhou ring. Then every square matrix over R is the sum of two tripotents and a nilpotent.*

Proof. In view of Lemma 2.3, R is isomorphic to R_1, R_2, R_3 or the products of these rings, where R_1 is a strongly nil-clean ring of bounded index, R_2 is a Yaqub ring and R_3 is a Bell ring.

Step 1. Let $A \in M_n(R_1)$. In view of [9, Corollary 6.8], there exist an idempotent $E \in M_n(R_1)$ and $W \in N(M_n(R_1))$ such that $A = E + W$.

Step 2. Let $A \in M_n(R_2)$, and let S be the subring of R_2 generated by the entries of A . That is, S is formed by finite sums of monomials of the form: $a_1 a_2 \cdots a_m$, where a_1, \dots, a_m are entries of A . Since R_2 is a commutative ring in which $3 = 0$, S is a finite ring in which $x = x^3$ for all $x \in S$. Thus, S is isomorphic to finite direct product of \mathbb{Z}_3 . As $A \in M_n(S)$, it follows by Lemma 2.1 that A is the sum of two tripotents and a nilpotent matrix over S .

Step 3. Let $A \in M_n(R_3)$, and let S be the subring of R_3 generated by the entries of A . Analogously, S is isomorphic to finite direct product of \mathbb{Z}_5 . As $A \in M_n(S)$, it follows by Lemma 2.2 that A is the sum of two tripotents and a nilpotent matrix over S .

Let $A \in M_n(R)$. We may write $A = (A_1, A_2, A_3)$ in $M_n(R_1) \times M_n(R_2) \times M_n(R_3)$, where $A_1 \in M_n(R_1)$, $A_2 \in M_n(R_2)$, $A_3 \in M_n(R_3)$. According to the preceding discussion, we easily complete the proof. \square

Corollary 2.5. *Let R in which for any $x \in R$, $x^5 = x$. Then every square matrix over R is the sum of two tripotents and a nilpotent.*

Proof. In view of [12, Theorem 2.11], every element in R is the sum of two tripotents and a nilpotent. But $N(R) = 0$, and so R is a Zhou ring. This completes the proof, by Theorem 2.4. \square

3. ZHOU NIL-CLEAN RINGS

The goal of this section is to explore matrix decompositions for Zhou nil-clean rings. The following lemma is crucial.

Lemma 3.1. *Let R be a ring. Then the following are equivalent:*

- (1) R is Zhou nil-clean.
- (2) $R/J(R)$ is a Zhou ring and $J(R)$ is nil.

Proof. (1) \Rightarrow (2) In view of [12, Theorem 2.11], $J(R)$ is nil and $R/J(R)$ has the identity $x = x^5$. By using [12, Theorem 2.11] again, every element in $R/J(R)$ is the sum of two tripotents and

a nilpotent \bar{w} . Clearly, $R/J(R)$ is reduced, and so $\bar{w} = \bar{0}$. Thus, $R/J(R)$ is a Zhou ring.

(2) \Rightarrow (1) Let $a \in R$. As $S := R/J(R)$ is a Zhou ring, it is Zhou nil-clean. It follows by [12, Theorem 2.11] that $\overline{a - a^5} \in N(R/J(R))$. As $J(R)$ is nil, we see that $a - a^5 \in N(R)$. According to [12, Theorem 2.11], R is Zhou nil-clean, as asserted. \square

We are ready to prove the following.

Theorem 3.2. *Let R be a 2-primal Zhou nil-clean ring. Then every square matrix over R is the sum of two tripotents and a nilpotent.*

Proof. In view of [12, Theorem 2.11], $R \cong S, T$ or $S \times T$, where S is a 2-primal strongly nil-clean ring, and T is a Zhou ring with $\frac{1}{2} \in T$.

Step 1. Let $A \in M_n(S)$. In view of [9, Theorem 6.1], A is the sum of an idempotent and a nilpotent.

Step 2. Let $A \in M_n(T)$. In view of [12, Theorem 2.11], $T/J(T)$ is commutative; and so $N(T) \subseteq J(T)$. Thus, $J(T) = N(T)$. Then $\bar{A} \in M_n(T/J(T))$. By virtue of Theorem 2.4, there exist tripotents $\bar{C}, \bar{D} \in M_n(T/J(T))$ and a nilpotent $\bar{W} \in M_n(T/J(T))$ such that $\bar{B} = \bar{C} + \bar{D} + \bar{W}$. Clearly, $M_n(R/J(R)) \cong M_n(R)/J(M_n(R))$. As R is 2-primal, it follows by [7, Theorem 10.21] that $J(M_n(T)) = M_n(J(T)) = M_n(N(T)) = M_n(N_*(T)) = N_*(M_n(T))$ is nil, where $N_*(T)$ is the Baer-McCoy radical of T . As $\frac{1}{2} \in M_n(T)$, by [12, Lemma 2.6], we may assume that $B, C \in M_n(T)$ are tripotents. Clearly, $W \in M_n(T)$ is nil. Thus, we can find $V \in J(M_n(T))$ such that $A = B + C + (W + V)$. Write $W^k = 0$. Then $(W + V)^k \in J(M_n(T))$, and so $(W + V)^{kl} = 0$ for some $l \in \mathbb{N}$. Therefore A is the sum of two tripotents and a nilpotent.

Let $A \in M_n(R)$. We may write $A = (A_1, A_2)$ in $M_n(S) \times M_n(T)$, where $A_1 \in M_n(S), A_2 \in M_n(T)$. Thus, by the preceding discussion, we obtain the result. \square

Corollary 3.3. *Let R be a commutative Zhou nil-clean ring. Then every square matrix over R is the sum of two tripotents and a nilpotent.*

Proof. This is obvious by Theorem 3.2, as every commutative ring is 2-primal. \square

Corollary 3.4. *Let R be a Zhou nil-clean ring, and let $A \in M_n(R)$ with central entries. Then A is the sum of two tripotents and a nilpotent.*

Proof. Let $C(R)$ be the center of R , and let $A \in C(R)$. In view of [12, Theorem 2.11], $a - a^2 \in N(R)$; and so $a - a^2 \in U(C(R))$. By using [12, Theorem 2.11] again, $C(R)$ is a commutative Zhou nil-clean. Since $A \in M_n(C(R))$, By Corollary 3.3, A is the sum of two tripotents and a nilpotent in $M_n(C(R))$. This completes the proof from $M_n(C(R)) \subseteq M_n(R)$. \square

Example 3.5. *Let $n \geq 2$ be an integer. If $n = 2^k 3^l 5^m$, then every square matrix over \mathbb{Z}_n is the sum of two tripotents and a nilpotent.*

Proof. As $n = 2^k 3^l 5^m$. It follows that $\mathbb{Z}_n \cong R_1 \times R_2 \times R_3$, where $R_1 = \mathbb{Z}_{2^k}$, $R_2 = \mathbb{Z}_{3^l}$ and $R_3 = \mathbb{Z}_{5^m}$. It is obvious that each $J(R_i)$ is nil and $R_1/J(R_1)$ is a Boolean ring, $R_2/J(R_2)$ is a Yaqub ring and $R_3/J(R_3)$ is a Bell ring. It follows by [12, Theorem 2.11], R is a commutative Zhou nil-clean ring. Therefore we complete the proof by Corollary 3.3. \square

Lemma 3.6. (see [9, Lemma 6.6]). *Let R be of bounded index. If $J(R)$ is nil, then $J(M_n(R))$ is nil for all $n \in \mathbb{N}$.*

Theorem 3.7. *Let R be a Zhou nil-clean ring of bounded index. Then every square matrix over R is the sum of two tripotents and a nilpotent.*

Proof. As in the proof of Theorem 3.3, $R = S \times T$, where S is strongly nil-clean of bounded index and T is a Zhou nil-clean ring with $\frac{1}{2} \in S$.

Step 1. Let $A \in M_n(S)$. In view of [9, Corollary 6.8], A is the sum of an idempotent and a nilpotent. Hence, it is the sum of two tripotents and a nilpotent.

Step 2. Let $A \in M_n(T)$. Then $\overline{A} \in M_n(T/J(T))$. In light of Lemma 3.1, $T/J(T)$ is a Zhou ring. By virtue of Theorem 2.4, there exist tripotents $\overline{C}, \overline{D} \in M_n(T/J(T))$ and a nilpotent $\overline{W} \in M_n(T/J(T))$ such that $\overline{A} = \overline{C} + \overline{D} + \overline{W}$. Clearly, $M_n(T/J(T)) \cong M_n(T)/J(M_n(T))$. In light of Lemma 3.6, $J(M_n(T))$ is nil. Since $\frac{1}{2} \in M_n(T)$, it follows by [12, Lemma 2.6] that we may assume that

$C, D \in M_n(R)$ are tripotents. Clearly, $W \in M_n(R)$ is nil. Thus, we can find $V \in J(M_n(R))$ such that $A = B + C + (W + V)$. Write $W^k = 0$. Then $(W + V)^{kl} = 0$ for some $l \in \mathbb{N}$. This shows that A is the sum of two tripotents and a nilpotent.

Let $A \in M_n(R)$. Then $A = (A_1, A_2)$ with $A_1 \in M_n(S)$ and $A_2 \in M_n(T)$. By the discussion in Step 1 and Step 2, we complete the proof. \square

Example 3.8. Let $n \geq 2$ be an integer, and let $S = T_s(\mathbb{Z}_n)(s \in \mathbb{N})$. If $n = 2^k 3^l 5^m$, then every square matrix over S is the sum of two tripotents and a nilpotent.

Proof. As in the proof of Example 3.5, \mathbb{Z}_n is a Zhou nil-clean ring. Let $A \in S$. Then $A - A^2 \in N(S)$, as its diagonal entries are nilpotent. It follows by [12, Theorem 2.11], S is a Zhou nil-clean ring. Clearly, S is of bounded index. Therefore we complete the proof, by Theorem 3.7. \square

REFERENCES

- [1] S. Breaz; G. Galugareanu; P. Danchev and T. Micu, Nil clean matrix rings, *Linear Algebra Appl.*, **439**(2013), 3115–3119.
- [2] H. Chen and M. Sheibani, Strongly 2-nil-clean rings, *J. Algebra Appl.*, **16**(2017), DOI: 10.1142/S021949881750178X.
- [3] H. Chen and M. Sheibani, Strongly weakly nil-clean rings, *J. Algebra Appl.*, **16**(2017), DOI: 10.1142/S0219498817502334.
- [4] P.V. Danchev and W.W. McGovern, Commutative weakly nil clean unital rings, *J. Algebra*, **425**(2015), 410–422.
- [5] A.J. Diesl, Nil clean rings, *J. Algebra*, **383**(2013), 197–211.
- [6] Y. Hirano and H. Tominaga, Rings in which every element is a sum of two idempotents, *Bull. Austral. Math. Soc.*, **37**(1988), 161–C164.
- [7] T.Y. Lam, *A First Course in Noncommutative Rings*, Springer-Verlag, New York, Heidelberg, Berlin, London, 2001.
- [8] M.T. Kosan; T.K. Lee and Y. Zhou, When is every matrix over a division ring a sum of an idempotent and a nilpotent? *Linear Algebra Appl.*, **450**2014, 7–12.
- [9] M.T. Kosan; Z. Wang and Y. Zhou, Nil-clean and strongly nil-clean rings, *J. Pure Appl. Algebra* (2015), <http://dx.doi.org/10.1016/j.jpaa.2015.07.009>.

- [10] Z. Ying; T. Kosan and Y. Zhou, Rings in which every element is a sum of two tripotents *Bull. Can. Math.*, January 2016, DOI: 10.4153/CMB-2016-009-0.
- [11] M.T. Kosan and Y. Zhou, On weakly nil-clean rings, *Front. Math. China*, June 2016, DOI: 10.1007 /s11464-016-0555-6.
- [12] Y. Zhou, Rings in which elements are sum of nilpotents, idempotents and nilpotents, *J. Algebra App.*, **16**(2017), DOI: 10.1142/S0219498818500093.

FACULTY OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE, SEMNAN UNIVERSITY, SEMNAN, IRAN

E-mail address: <m.sheibani1@gmail.com>

DEPARTMENT OF MATHEMATICS, HANGZHOU NORMAL UNIVERSITY, HANGZHOU, CHINA

E-mail address: <huanyinchen@aliyun.com>